Approximate Solution of the Von Mises Boundary-Layer Equation

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Theme

A MODIFIED Oseen approximation has been developed for solving the laminar boundary-layer equations and, in particular, for computing the shearing stress downstream of velocity profiles of arbitrary shape and for external flows with constant pressure gradient parameter. The approximate method of solution is outlined and several examples, showing a very good agreement with other methods, are presented.

Content

The incompressible laminar boundary layer is considered here. Introducing the following dimensionless variables

$\bar{x}=x/L$	streamwise coordinate
$\bar{y}=y\sqrt{Re}/L$	normal coordinate
$\bar{\psi}=\psi\sqrt{Re}/U_\infty L$	stream function
$\bar{u}=u/U_\infty$	streamwise velocity component
$\bar{U}=U/U_\infty$	external flow velocity
$\bar{\tau}=\tau L/\mu U_\infty\sqrt{Re}$	shearing stress on the wall

where L, U_∞, $Re=\rho U_\infty L/\mu$ are reference length, velocity, and Reynolds number. Dropping from now on, for the sake of simplicity, the bars denoting the dimensionless quantities, the momentum equation can be written in Von Mises coordinates[1] in the form

$$\partial g/\partial x = u\partial^2 g/\partial\psi^2 \tag{1}$$

with $g=(u^2-U^2)/2$. The shearing stress on the wall is given by

$$\tau = \partial g/\partial\psi)_w$$

Using the same arguments adopted by Lewis and Carrier[2] and in a modified form by Libby and Schetz,[3] Eq. (1) is linearized by replacing the velocity $u(x,\psi)$ with an "effective velocity" $\tilde{u}(x)$; introducing then an auxiliary variable ξ, defined by

$$\mathrm{d}\xi/\mathrm{d}x = \tilde{u}(x)$$

we obtain the equation

$$\partial g/\partial\xi = \partial^2 g/\partial\psi^2 \tag{2}$$

Equation (2) can be easily solved in closed form[4] and, in particular, the shearing stress on the wall results in

$$\tau(\xi) = \frac{1}{2\sqrt{\pi\xi}}\left[\int_0^\infty e^{-\psi^2/4\xi}\frac{\mathrm{d}u_0^2}{\mathrm{d}\psi}\,\mathrm{d}\psi + \int_0^\xi \frac{\mathrm{d}U^2/\mathrm{d}t}{\sqrt{1-t/\xi}}\,\mathrm{d}t\right] \tag{3}$$

where $u_0(\psi)$ is the initial velocity profile.

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In order to obtain the expression of $\tau(x)$ in the physical plane, it is then necessary to determine the "effective velocity" or, which is the same, the relationship between ξ and x. An exact relationship can be established for the class of the similar profiles, by comparison with the well-known solutions of the Falkner-Skan equation.[5] For each value of the pressure gradient parameter β, a family of exact solutions exists, corresponding to different initial conditions (initial thickness and shearing stress). The analysis leads to the interesting result that a correlation between ξ and x can be established, which does not depend explicitly on the initial conditions, but only on the value of β and on the local values of U and τ. The computations have been carried out for several values of β and the results are shown in Fig. 1, where H and K are reference values defined in the full paper.

The fact that the relationship between ξ and x does not depend on the initial conditions suggests the hypothesis that the same correlation might be used also for initial velocity profiles of arbitrary shape (but still with constant β). With this assumption, the shearing stress can be easily computed, using Eq. (3) and the above mentioned correlation. In particular, for $\beta=0$, the relationship between ξ and x can be written in the form

$$\frac{\xi}{x} = 1.101612\, I^{1/2} + 0.063277 I - 0.251402 I^{3/2} + \epsilon \tag{4}$$

where

$$I = \frac{1}{2}\int_0^\infty e^{-\psi^2/4\xi}\frac{\mathrm{d}u_0^2}{\mathrm{d}\psi}\,\mathrm{d}\psi \quad \text{and} \quad |\epsilon| < 0.0008$$

Moreover, in this case the calculations are extremely simple, because the second integral in Eq. (3) vanishes: $\tau(\xi)$ is computed from Eq. (3) and then $x(\xi)$ is obtained from Eq. (4).

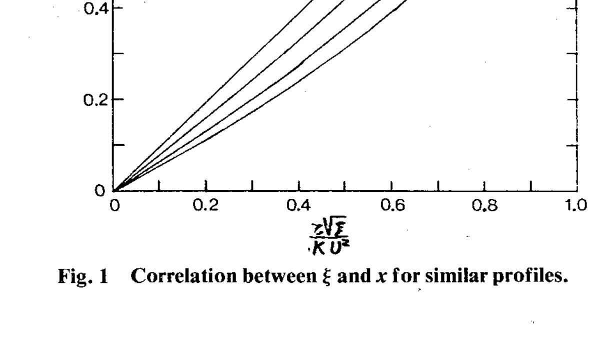

Fig. 1 Correlation between ξ and x for similar profiles.

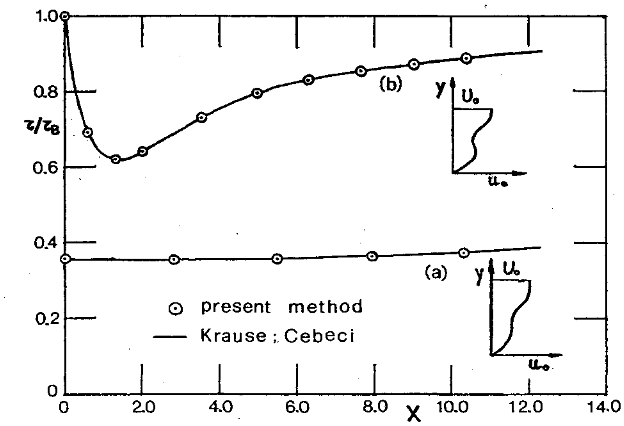

Fig. 2 Shearing stress downstream of boundary-layer-jet interaction (a) and of boundary-layer-wake interaction (b), $\beta = 0$.

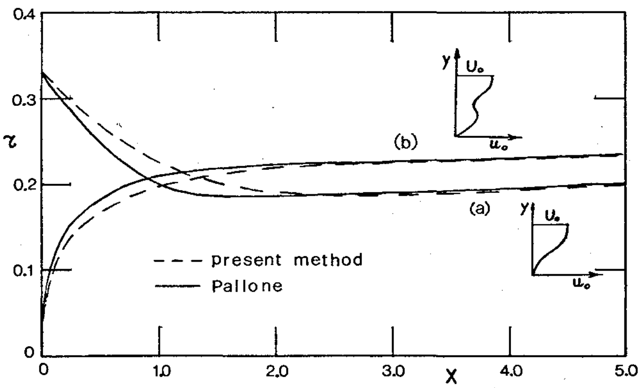

Fig. 3 Shearing stress downstream of an injection region (a) and of a boundary-layer-wake interaction (b), $\beta = 1$.

For $\beta \neq 0$ the computation is more complicated, because $U(\xi)$ is not known in advance; in this case, the solution is obtained with a step-by-step procedure.

Several numerical examples have been carried out, in order to test the validity of the method. Two examples deal with the boundary layer along a flat plate ($\beta = 0$); the shapes of the initial profiles are given in the full paper and correspond, respectively, to a boundary-layer-jet interaction and to a boundary-layer-wake interaction. The solutions are plotted in Fig. 2, where the ordinates represent the ratio

$$\frac{\tau}{\tau_B} = \frac{\tau\sqrt{1+x}}{0.33206}$$

and τ_B is the shearing stress due to a Blasius profile with origin at $x = -1$. As it appears, the results show an excellent agreement with the exact numerical solutions obtained by Krause[6] and by Cebeci and Smith.[7]

Two other examples, corresponding to an external flow with $\beta = 1$ and given by $U = 1 - 0.02x$, are shown in Fig. 3. The initial conditions considered here are the injection profile and the boundary-layer-wake interaction profile (the shapes are given in the full paper). No exact numerical solutions were available to the author; in order to make a comparison, the computations also have been performed using the strip method of Pallone[8] (with six strips). It appears that, even though the asymptotic behaviors of the two methods agree quite well, there are some differences in the initial part; however, this discrepancy also could be due to the fact that the strip method is approximate; an exact numerical solution would be necessary to determine the degree of approximation of the present method.

Some preliminary results show that it is possible to extend the method to the more general case of flows with arbitrary pressure gradient; in this case, however, the procedure becomes much more involved and the greatest advantage of the method, i.e., its simplicity, is lost. Therefore, the present approximation appears to be most useful for the study of the boundary layer in wedge flows with arbitrary initial conditions.

References

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